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Global existence and maximal regularity of solutions of gradient systems

Sahbi Boussandel [a,b,*]

[a] Faculté des Sciences de Bizerte, Département de Mathématiques, 7021 Jarzouna Bizerte, Tunisia
[b] Université Paul Verlaine – Metz, Laboratoire de Mathématiques et Applications de Metz et CNRS, UMR 7122, Bât. A, Île de Saulcy, 57045 Metz Cedex 1, France

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ABSTRACT

In this article, we use a Galerkin method to prove a maximal regularity result for the following abstract gradient system

$$\begin{cases} u'(t) + \nabla_{g(t)} E\big(u(t)\big) = f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = u_0. \end{cases}$$

This abstract result is applied to nonlinear diffusion equations and to nondegenerate quasilinear parabolic equations with nonlocal coefficients.

1. Introduction

In this work, we prove a global existence and maximal regularity result of solutions of an abstract gradient system.

In order to formulate the problem, let V be a real Banach space and let H be a real Hilbert space such that V is densely and compactly embedded in H. Let $E : V \to \mathbb{R}$ be a continuously differentiable function. Consider the gradient system

$$\begin{cases} u'(t) + \nabla_g E\big(u(t)\big) = f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{1}$$

* Address for correspondence: Faculté des Sciences de Bizerte, Département de Mathématiques, 7021 Jarzouna Bizerte, Tunisia.

E-mail address: boussand@univ-metz.fr.

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where $u_0 \in V$, $f \in L^2(0,T;H)$ are given and $\nabla_g E$ denotes the gradient of E with respect to some metric g.

Under suitable conditions on V, g and E, we prove a maximal regularity result for system (1), in the sense that, for every $u_0 \in V$ and every $f \in L^2(0,T;H)$, there exists $u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$ such that $u(t) \in D(\nabla_g E)$ for almost every $t \in (0,T)$, which is a solution of system (1). In particular, the two terms on the left-hand side of the above system have the same regularity as the right-hand side term.

Note that every function in $W^{1,2}(0,T;H)$ is continuous with values in H, which implies that the initial condition in system (1) makes sense.

Several authors have studied abstract gradient systems in various frameworks. In the case where the metric g is constant, that is, independent of u, we mention first J.-L. Lions [13, chapitre 2] who proved a maximal regularity result in V', that is, for right-hand sides $f \in L^{p'}(0,T;V')$ and initial values in H. The theory of subgradients gives maximal regularity results in H and includes the results obtained here if g is constant and E is convex; two references are [6] and [17]. However, in both approaches it is not clear how to include general metrics; concrete examples in which it is necessary to consider general metrics arise, for example in geometric evolution problems, that is, in the evolution of curves and surfaces. Usually such problems fall within the theory of quasilinear evolution problems and can be solved by maximal regularity results for linear problems and fixed point theorems; see [2,12] and [14]; however, these approaches seem to fail if degenerate operators (p-Laplace operators) are involved. It is possible that our result is covered by the theory of gradient systems on general metric spaces (see [3]), however if it is, our approach by space discretization (Faedo–Galerkin approximation) is certainly not included, but probably important from the point of view of numerical analysis. Moreover, our result covers also nonautonomous problems in which the metric g depends on time and is only measurable in time.

We apply our result to the partial differential equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(t,\cdot,u)\Delta_p u = f & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \end{cases} \tag{2}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\big(a(\cdot,u)\nabla u\big) = f & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } \quad (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \end{cases} \tag{3}$$

in which the coefficient m is measurable with respect to first two variables and continuous with respect to the third variable and the coefficient a is measurable with respect to the first variable and satisfies a weak continuity assumption with respect to the second variable (see Examples 1 and 3).

Eq. (2) has recently been considered by W. Arendt and R. Chill [1] in the special case $p = 2$. Like in [1], we can in general not prove uniqueness, neither for the abstract system (1), nor for the problems (2) and (3). The problem of uniqueness is open.

2. The main result

Let V be a real, separable and reflexive Banach space with norm $\|\cdot\|_V$, let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and induced norm $\|\cdot\|_H$, and assume that V is densely and continuously embedded into H. The duality bracket between the dual space V' and V is denoted by $\langle\cdot,\cdot\rangle_{V',V}$. Let $\mathrm{Inner}(H)$ be the set of all inner products on H which are equivalent to the fixed one on H.

Definition 1. A function $g : V \to \mathrm{Inner}(H)$ is a metric if it satisfies the weak continuity condition

$$u_n \rightharpoonup u \quad \text{in } V \quad \Rightarrow \quad \langle w, v\rangle_{g(u_n)} \to \langle w, v\rangle_{g(u)} \quad \text{for every } v, w \in H.$$

We denote by $\langle\cdot,\cdot\rangle_{g(u)}$ the inner product $g(u)$ at a point $u \in V$ and by $\|\cdot\|_{g(u)}$ the norm associated with this inner product.

Let $E : V \to \mathbb{R}$ be a continuously differentiable function. We denote by E' the Fréchet-derivative of E. Recall that the Fréchet-derivative of E at a point u is an element of V'.

Definition 2. We define the gradient of E in H with respect to the metric g by

$$\begin{aligned} &D(\nabla_g E) = \{u \in V\colon \exists w \in H,\ \forall v \in V,\ E'(u)v = \langle w, v\rangle_{g(u)}\},\\ &\nabla_g E(u) = w. \end{aligned}$$

Note that since V is densely embedded into H, the element $\nabla_g E(u)$ is uniquely determined.

Throughout the rest of this article we actually consider time dependent metrics, that is, given $T > 0$, we consider a function $g : [0, T] \times V \to \mathrm{Inner}(H)$ such that $g(t,\cdot)$ is a metric for every $t \in [0, T]$. We define the gradient of the function E with respect to the metric $g(t,\cdot)$ like in the last definition and we denote it accordingly by $\nabla_{g(t)} E$.

Let $T > 0$, $f \in L^2(0, T; H)$ and $u_0 \in V$. We consider the problem of finding u such that

$$\begin{cases} u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V), \quad u(t) \in D(\nabla_{g(t)}E) & \text{for a.e. } t \in (0,T),\\ u'(t) + \nabla_{g(t)} E\big(u(t)\big) = f(t) & \text{for a.e. } t \in (0,T),\\ u(0) = u_0. \end{cases} \tag{4}$$

We call the evolution equation in (4) an *abstract gradient system*.

By using the density of V in H, the separability of V and the definition of the gradient $\nabla_{g(t)} E$, problem (4) is equivalent to the variational problem of finding u such that

$$\begin{cases} u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V),\\ \big\langle u'(t), v\big\rangle_{g(t,u(t))} + E'\big(u(t)\big)v = \big\langle f(t), v\big\rangle_{g(t,u(t))} \quad \text{for every } v \in V, \text{ and for a.e. } t \in (0,T),\\ u(0) = u_0. \end{cases} \tag{5}$$

Definition 3. A function $E : V \to \mathbb{R}$ is H-elliptic if there exists $\omega \in \mathbb{R}$ such that the function $E_\omega : V \to \mathbb{R}$, $u \to E(u) + \frac{\omega}{2}\|u\|_H^2$ is

(a) coercive: for every $R > 0$ there exists $C_R > 0$ such that for every $u \in V$ if $E_\omega(u) \leqslant R$ then $\|u\|_V \leqslant C_R$,
(b) convex: for every $u, v \in V$, $t \in [0,1]$, $E_\omega((1-t)u + tv) \leqslant (1-t)E_\omega(u) + tE_\omega(v)$.

The main result of this article is the following theorem:

Theorem 4. *Suppose that V is a reflexive, separable Banach space which is compactly and densely embedded into H. Suppose that E is an H-elliptic, continuously differentiable function such that the derivative $E' : V \to V'$ maps bounded sets into bounded sets.*

Let $T > 0$. Suppose further that $g(t,\cdot)$ is a metric for every $t \in [0,T]$, and for every $v, w \in H$, $u \in V$, the function $t \to \langle v, w\rangle_{g(t,u)}$ is measurable on $[0,T]$. Suppose in addition that there exist two constants $c_1, c_2 > 0$ such that for every $u \in V$, every $v \in H$ and for every $t \in [0,T]$

$$c_1\|v\|_H \leqslant \|v\|_{g(t,u)} \leqslant c_2\|v\|_H. \tag{6}$$

Suppose finally that the metric $g(t,\cdot)$ is continuous in the sense that

$$\left.\begin{array}{ll} u_n \rightharpoonup u & \text{in } W^{1,2}(0,T;H), \\ u_n \overset{\text{weak}^*}{\to} u & \text{in } L^\infty(0,T;V), \\ v_n \rightharpoonup v & \text{in } L^2(0,T;H), \\ w \in L^2(0,T;H) & \end{array}\right\} \quad \Rightarrow \quad \int_0^T \langle v_n, w\rangle_{g(t,u_n)}\,dt \to \int_0^T \langle v, w\rangle_{g(t,u)}\,dt. \tag{7}$$

Then, for every $u_0 \in V$ and every $f \in L^2(0,T;H)$, problem (4) *admits a solution.*

Remark 5. We remark that Theorem 4 is an L^2-maximal regularity result for the nonlinear problem (4) in the sense that for every $f \in L^2(0,T;H)$ and every $u_0 \in V$, problem (4) admits a solution u (however not necessarily unique) such that the two members u' and $\nabla_{g(\cdot)}E(u)$ of the left-hand side of the evolution equation in problem (4) belong also to $L^2(0,T;H)$. Compare with the definition of the L^2-maximal regularity of linear problems in [4,9,11].

Remark 6. If $g(t,\cdot)$ is a metric for every $t \in [0,T]$, and if we suppose that assumption (6) holds then

$$\left.\begin{array}{ll} u_n \to u & \text{in } V, \\ v_n \to v & \text{in } H, \\ w_n \to w & \text{in } H \end{array}\right\} \quad \Rightarrow \quad \langle v_n, w_n\rangle_{g(t,u_n)} \to \langle v, w\rangle_{g(t,u)}, \quad \text{for every } t \in [0,T].$$

In fact, this assertion is a simple consequence of the following inequality

$$\begin{aligned} \left|\langle v_n, w_n\rangle_{g(t,u_n)} - \langle v, w\rangle_{g(t,u)}\right| &\leqslant \left|\langle v, w\rangle_{g(t,u_n)} - \langle v, w\rangle_{g(t,u)}\right| \\ &\quad + \left|\langle v_n, w_n - w\rangle_{g(t,u_n)}\right| + \left|\langle v_n - v, w\rangle_{g(t,u_n)}\right|. \end{aligned}$$

Remark 7. Let $u : [0,T] \to V$, $v : [0,T] \to H$ and $w : [0,T] \to H$ be three measurable functions and assume that $g(t,\cdot)$ is a metric for every $t \in [0,T]$ which satisfies the measurability condition in Theorem 4. Then $t \to \langle v(t), w(t)\rangle_{g(t,u(t))}$ is a measurable function on $[0,T]$. In fact, by using the measurability condition on g in Theorem 4, this assertion holds for step functions in V and H. The general case is a consequence of the definition of measurable functions as the pointwise limit of step functions, and Remark 6.

This result shows that the terms which appear under integral sign in the continuity assumption (7) are measurable. This result can also be used later in the proof of Theorem 4.

3. Proof of Theorem 4

To prove Theorem 4, we need the following two lemmas.

Lemma 8. *Assume that the embedding $V \hookrightarrow H$ is compact. Then the embedding*

$$W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \hookrightarrow C\big([0,T];H\big)$$

is compact, too.

Proof. We refer the reader to [18, Corollary 4, p. 85] for the proof of this lemma. □

Lemma 9. *Let $E : V \to \mathbb{R}$ be a continuously differentiable and convex function. Then $E' : V \to V'$ is a monotone operator, that is, for every $u, v \in V$ one has*

$$\langle E'(u) - E'(v), u - v \rangle_{V',V} \geqslant 0.$$

Proof. For the proof of this lemma, see [13, Proposition 1.1, p. 158]. □

Proof of Theorem 4. To prove the theorem, we use a Galerkin approximation.

Part 1 *(Formulation of the finite-dimensional approximating problems).* Let (w_n) be any sequence in V such that $\operatorname{span}\{w_n\colon n \geqslant 1\}$ is dense in V; such a sequence exists since V is a separable space. For every $m \in \mathbb{N}$, we put

$$V_m = \operatorname{span}\{w_n,\ 1 \leqslant n \leqslant m\},$$

and we choose $u_0^m \in V_m$ such that

$$u_0 = \lim_{m\to\infty} u_0^m \quad \text{in } V.$$

There indeed exists such a sequence (u_0^m) since $\bigcup_m V_m$ is dense in V and the sequence (V_m) is increasing.

For every $m \in \mathbb{N}$, we consider the variational problem of finding u_m such that

$$\begin{cases} u_m \in W^{1,2}_{\mathrm{loc}}\big([0,T); V_m\big), \\ \langle u_m'(t), v\rangle_{g(t,u_m(t))} + E'\big(u_m(t)\big)v = \langle f(t), v\rangle_{g(t,u_m(t))} \quad \text{for every } v \in V_m \text{ and a.e. } t \in (0,T), \\ u_m(0) = u_0^m. \end{cases} \tag{8}$$

Problem (8) is equivalent to the problem of finding a solution u_m of the following ordinary differential equation

$$\begin{cases} u_m \in W^{1,2}_{\mathrm{loc}}\big([0,T); V_m\big), \\ u_m'(t) + \nabla_{g_m(t)} E_m\big(u_m(t)\big) = P_m\big(t, u_m(t)\big) f(t), \quad \text{a.e. } t \in (0,T), \\ u_m(0) = u_0^m, \end{cases} \tag{9}$$

where E_m is the restriction of E to V_m, $g_m : [0,T] \times V_m \to \mathrm{Inner(V_m)}$ is the function defined for every $u \in V_m$, for every $t \in [0,T]$ and for every $v, w \in V_m$ by

$$\langle v, w\rangle_{g_m(t,u)} = \langle v, w\rangle_{g(t,u)},$$

and $P_m(t,u_m) : H \to H$ is the orthogonal projection from H onto V_m with respect to the inner product $\langle\cdot,\cdot\rangle_{g(t,u_m)}$. By the Riesz–Fréchet theorem and since V_m is finite-dimensional, for every $m \in \mathbb{N}$, every $u \in V_m$ and for every $t \in [0,T]$, the gradient $\nabla_{g_m(t)} E_m(u)$ exists and belongs to V_m.

In order to obtain existence of maximal solutions for problem (9), we check that the function $F : (0,T) \times V_m \to V_m$, $(t,u) \to \nabla_{g_m(t)} E_m(u) - P_m(t,u) f(t)$ satisfies the Carathéodory conditions:

(a) $F(\cdot, u)$ is measurable for every $u \in V_m$,
(b) $F(t, \cdot)$ is continuous for almost every $t \in (0,T)$, and
(c) for every $(t_0, u_0) \in (0,T) \times V_m$, there exist $\alpha, r > 0$ and $m \in L^1(t_0, t_0 + \alpha)$ such that $\|F(t,u)\| \leqslant m(t)$ for almost every $t \in (t_0, t_0 + \alpha)$ and every $u \in V_m$ such that $\|u - u_0\|_{V_m} < r$.

For every $u \in V_m$ and for every $t \in [0, T]$, we consider the operator $Q_m(t, u) \in \mathcal{L}(V_m)$ defined for every $v, w \in V_m$ by

$$\langle Q_m(t, u)v, w\rangle_H = \langle v, w\rangle_{g_m(t,u)} = \langle v, w\rangle_{g(t,u)}. \tag{10}$$

For every $u, v \in V_m$ and for every $t \in [0, T]$ one has

$$\begin{aligned}\langle Q_m(t, u)P_m(t, u)f(t), v\rangle_H &= \langle P_m(t, u)f(t), v\rangle_{g(t,u)}\\ &= \langle f(t), P_m(t, u)v\rangle_{g(t,u)}\\ &= \langle f(t), v\rangle_{g(t,u)}.\end{aligned} \tag{11}$$

By Remark 7, for every $u, v \in V_m$, the function $t \to \langle f(t), v\rangle_{g(t,u)}$ is measurable on $(0, T)$. Using equality (11), we obtain that for every $u, v \in V_m$, the function $t \to \langle Q_m(t, u)P_m(t, u)f(t), v\rangle_H$ is measurable on $(0, T)$. This proves that for every $u \in V_m$, the function $t \to Q_m(t, u)P_m(t, u)f(t)$ is weakly measurable and then measurable on $(0, T)$ since V_m is a finite-dimensional space (we refer the reader to [10] for more details about measurable and weakly measurable functions). From the definition of the operator $Q_m(t, u)$ and since $\mathcal{L}(V_m)$ is a finite-dimensional space, we obtain that for every $u \in V_m$, the function $t \to Q_m(t, u)$ is measurable on $[0, T]$. Using the fact that the operator $Q_m(t, u)$ is invertible and that taking the inverse is a homeomorphism in the set of all invertible operators, we deduce that for every $u \in V_m$, the function $t \to P_m(t, u)f(t)$ is measurable on $(0, T)$.

In addition, for every $u, v \in V_m$ and for every $t \in [0, T]$

$$\begin{aligned}\langle \nabla_{g_m(t)}E_m(u), v\rangle_H &= \langle \nabla_{g_m(t)}E_m(u), Q_m(t, u)Q_m(t, u)^{-1}v\rangle_H\\ &= \langle \nabla_{g_m(t)}E_m(u), Q_m(t, u)^{-1}v\rangle_{g_m(t,u)}\\ &= E'(u)\big(Q_m(t, u)^{-1}v\big).\end{aligned}$$

Since $t \to Q_m(t, u)^{-1}v$ is measurable on $[0, T]$ and $E'(u)$ is a continuous linear operator on V_m, we deduce that for every $u \in V$, the function $t \to \nabla_{g_m(t)}E_m(u)$ is measurable on $[0, T]$. Hence, the Carathéodory condition (a) is satisfied.

By (10), the operator $Q_m(t, \cdot)$ is continuous on V_m for every $t \in [0, T]$. This yields that $Q_m(t, \cdot)^{-1}$ and hence $\nabla_{g_m(t)}E_m(\cdot)$ are continuous on V_m for every $t \in [0, T]$.

Similarly, by using (11), we see that $P_m(t, \cdot)f(t)$ is continuous on V_m for every $t \in [0, T]$. This proves that the Carathéodory condition (b) is satisfied.

Finally, it remains to check that the Carathéodory condition (c) is satisfied. Since V_m is finite-dimensional and any two norms on V_m are equivalent, it suffices to estimate $\|\nabla_{g_m(t)}E_m(u)\|_H$ and $\|P_m(t, u)f(t)\|_H$.

Let $u_0 \in V_m$, $r > 0$ and $u \in V_m$ such that $\|u - u_0\|_{V_m} < r$. Note that by (6) and the definition of Q_m we obtain $\|Q_m(t, u)\| \leqslant c_2^2$ with respect to the H-norm. Since $\langle v, w\rangle_H = \langle Q_m(t, u)^{-1}v, w\rangle_{g(t,u)}$, one has for every $v \in V_m$

$$\begin{aligned}\big\|Q_m(t, u)^{-1}v\big\|_H &\leqslant \frac{1}{c_1}\big\|Q_m(t, u)^{-1}v\big\|_{g(t,u)}\\ &= \frac{1}{c_1}\sup_{\|w\|_{g(t,u)}\leqslant 1}\big|\langle v, w\rangle_H\big|\\ &\leqslant \frac{1}{c_1^2}\|v\|_H,\end{aligned}$$

hence $\|Q_m(t,u)^{-1}\| \leqslant \frac{1}{c_1^2}$ with respect to the H-norm. Now the bounds for condition (c) follow easily from the formulas used in part (a).

Hence, by [8, Theorem 4.1, Chapter 1], problem (9) admits a maximal solution $u_m \in W^{1,2}_{\mathrm{loc}}([0,T_m); V_m)$ in the sense that either $T_m = T$, or $T_m < T$ and the solution u_m cannot be extended to any larger interval. For every $m \in \mathbb{N}$, let u_m be a maximal solution of (9).

Part 2 *(Bounds for the solutions $\boldsymbol{u_m}$ of the approximating problems).* We take $v = u_m'$ in Eq. (8). Then we integrate over the interval $(0,t)$, for $t \in (0, T_m)$, and we obtain

$$\int_0^t \|u_m'(s)\|^2_{g(s,u_m(s))}\,ds + E\big(u_m(t)\big) - E\big(u_0^m\big) = \int_0^t \big\langle f(s), u_m'(s)\big\rangle_{g(s,u_m(s))}\,ds.$$

Since $u_0^m \to u_0$ in V, and since E is continuous, we have $\lim_{m\to\infty} E(u_0^m) = E(u_0)$ and in particular the sequence $(E(u_0^m))$ is bounded. Hence, there exists a constant $c_3 > 0$ which is independent of m and t such that

$$\int_0^t \|u_m'(s)\|^2_{g(s,u_m(s))}\,ds + E\big(u_m(t)\big) \leqslant c_3 + \int_0^t \big\langle f(s), u_m'(s)\big\rangle_{g(s,u_m(s))}\,ds.$$

We employ assumption (6) in order to obtain

$$c_1^2 \int_0^t \|u_m'(s)\|^2_H\,ds + E\big(u_m(t)\big) \leqslant c_3 + c_2^2 \int_0^t \|f(s)\|_H \|u_m'(s)\|_H\,ds.$$

By using Young's inequality, we deduce that there exists a constant $c_4 > 0$ which is independent of m and t such that

$$\frac{c_1^2}{2} \int_0^t \|u_m'(s)\|^2_H\,ds + E\big(u_m(t)\big) \leqslant c_4.$$

Let $\omega \geqslant 0$ such that E_ω is convex and coercive. Then the preceding inequality can be rewritten as

$$\frac{c_1^2}{2} \int_0^t \|u_m'(s)\|^2_H\,ds + \|u_m(t)\|^2_H + E_\omega\big(u_m(t)\big) \leqslant c_4 + \frac{\omega+2}{2}\|u_m(t)\|^2_H. \tag{12}$$

Moreover, there exist two constants c_5, $c_6 > 0$ which are independent of m and t such that the following estimate holds

$$\begin{aligned}\|u_m(t)\|^2_H &= \|u_m(0)\|^2_H + \int_0^t \frac{d}{ds}\|u_m(s)\|^2_H\,ds = \|u_m(0)\|^2_H + 2\int_0^t \big\langle u_m'(s), u_m(s)\big\rangle_H\,ds \\ &\leqslant c_5 + \frac{c_1^2}{2(\omega+2)} \int_0^t \|u_m'\|^2_H\,ds + c_6 \int_0^t \|u_m\|^2_H\,ds.\end{aligned}$$

By combining this last inequality with inequality (12), we have the existence of two constants c_7, $c_8 > 0$ which are independent of m and t such that

$$\frac{c_1^2}{4}\int_0^t \left\|u_m'(s)\right\|_H^2\,ds + \left\|u_m(t)\right\|_H^2 + E_\omega\big(u_m(t)\big) \leqslant c_7 + c_8\int_0^t \left\|u_m(s)\right\|_H^2\,ds.$$

Since E_ω is continuous, convex and coercive, E_ω is bounded from below (in fact, since V is reflexive, E_ω even attains a minimum). Hence, there exists a constant $c_9 > 0$ which is independent of m and t such that the last estimate implies

$$\frac{c_1^2}{4}\int_0^t \left\|u_m'(s)\right\|_H^2\,ds + \left\|u_m(t)\right\|_H^2 \leqslant c_9 + c_8\int_0^t \left\|u_m(s)\right\|_H^2\,ds. \tag{13}$$

It follows that

$$\left\|u_m(t)\right\|_H^2 \leqslant c_9 + c_8\int_0^t \left\|u_m(s)\right\|_H^2\,ds.$$

By Gronwall's lemma, there exists a positive constant c_{10} such that

$$\sup_{m\in\mathbb{N}}\sup_{t\in(0,T_m)} \left\|u_m(t)\right\|_H^2 \leqslant c_{10}.$$

We return to inequality (12), we employ this last estimate, and we have the existence of a constant c_{11} which is independent of m and t such that

$$\frac{c_1^2}{2}\int_0^t \left\|u_m'(s)\right\|_H^2\,ds + \left\|u_m(t)\right\|_H^2 + E_\omega\big(u_m(t)\big) \leqslant c_{11}. \tag{14}$$

This implies

$$\sup_{m\in\mathbb{N}}\sup_{t\in(0,T_m)} E_\omega\big(u_m(t)\big) \leqslant c_{11}.$$

Using the fact that E_ω is coercive, this implies that there exists a constant $c_{12} > 0$ such that

$$\sup_{m\in\mathbb{N}}\sup_{t\in(0,T_m)} \left\|u_m(t)\right\|_V \leqslant c_{12}.$$

Using again the fact that E_ω is bounded from below, we deduce from inequality (14) that

$$\sup_{m\in\mathbb{N}} \|u_m\|_{W^{1,2}(0,T_m;H)} < \infty.$$

Since $T_m \leqslant T$ is finite, this implies that for each $m \in \mathbb{N}$ the function u_m' is integrable on $[0, T_m)$. Hence, u_m extends to a continuous function on the closed interval $[0, T_m]$, and [8, Theorem 1.1, Chapter 2] and the definition of the maximal solution imply that this is only possible if $T_m = T$, that is, the solutions u_m are global.

From the preceding two inequalities we obtain

$$(u_m) \quad \text{is bounded in } W^{1,2}(0,T;H) \cap L^\infty(0,T;V).$$

By assumption, E' maps bounded sets into bounded sets, so that the boundedness of (u_m) in $L^\infty(0,T;V)$ implies that

$$\big(E'(u_m)\big) \quad \text{is bounded in } L^\infty\big(0,T;V'\big).$$

Part 3 *(Extracting a convergent subsequence).* Since V is a reflexive space, the space $L^\infty(0,T;V)$ is isometrically isomorphic to the dual space of $L^1(0,T;V')$. Moreover $L^1(0,T;V')$ is a separable space since V' is a separable space. Then by the Banach–Alaoglu theorem and by Lemma 8, we can extract from (u_m) a sequence (which we denote again by (u_m)) such that

$$u_m \rightharpoonup u \qquad \text{in } W^{1,2}(0,T;H), \tag{15}$$

$$u_m \overset{w^*}{\to} u \qquad \text{in } L^\infty(0,T;V), \tag{16}$$

$$u_m \to u \qquad \text{in } C\big([0,T];H\big), \quad \text{and} \tag{17}$$

$$E'(u_m) \overset{w^*}{\to} \chi \quad \text{in } L^\infty\big(0,T;V'\big). \tag{18}$$

Part 4 *(Showing that the limit **u** is a solution of problem (4)).* Let $w \in V_m$ and $\varphi \in L^2(0,T)$. Then for every $n \geqslant m$ we have from Eq. (8)

$$\int_0^T \big\langle u_n', \varphi(t)\, w\big\rangle_{g(t,u_n)}\, dt + \int_0^T E'(u_n)\varphi(t) w\, dt = \int_0^T \big\langle f(t)\varphi(t), w\big\rangle_{g(t,u_n)}\, dt.$$

Letting $n \to \infty$ in this last equality, and using (15), (18) and the continuity assumption (7), we obtain

$$\int_0^T \big\langle u', \varphi(t) w\big\rangle_{g(t,u)}\, dt + \int_0^T \big\langle \chi, \varphi(t) w\big\rangle_{V',V}\, dt = \int_0^T \big\langle f(t), \varphi(t) w\big\rangle_{g(t,u)}\, dt. \tag{19}$$

Using the fact that $\{\varphi(\cdot) w,\ w \in \bigcup_m V_m,\ \varphi \in L^2(0,T)\}$ spans a dense subspace of $L^2(0,T;V)$, equality (19) implies for every $v \in L^2(0,T;V)$

$$\int_0^T \big\langle u', v\big\rangle_{g(t,u)}\, dt + \int_0^T \langle \chi, v\rangle_{V',V}\, dt = \int_0^T \big\langle f(t), v\big\rangle_{g(t,u)}\, dt. \tag{20}$$

We take $v = u \in L^2(0,T;V)$ in equality (20) and we obtain

$$\int_0^T \big\langle u', u\big\rangle_{g(t,u)}\, dt + \int_0^T \langle \chi, u\rangle_{V',V}\, dt = \int_0^T \big\langle f(t), u\big\rangle_{g(t,u)}\, dt. \tag{21}$$

We have also from Eq. (8)

$$\int_0^T E'(u_n)u_n\,dt = \int_0^T \langle f(t), u_n\rangle_{g(t,u_n)}\,dt - \int_0^T \langle u_n', u_n\rangle_{g(t,u_n)}\,dt. \tag{22}$$

The continuity assumption (7) and (15) imply

$$\int_0^T \langle f(t), u_n\rangle_{g(t,u_n)}\,dt \to \int_0^T \langle f(t), u\rangle_{g(t,u)}\,dt. \tag{23}$$

One has the following equality

$$\int_0^T \langle u_n', u_n\rangle_{g(t,u_n)}\,dt = \int_0^T \langle u_n', u\rangle_{g(t,u_n)}\,dt + \int_0^T \langle u_n', u_n - u\rangle_{g(t,u_n)}\,dt. \tag{24}$$

Using again the continuity assumption (7) and (15), we obtain

$$\int_0^T \langle u_n', u\rangle_{g(t,u_n)}\,dt \to \int_0^T \langle u', u\rangle_{g(t,u)}\,dt. \tag{25}$$

By using the Cauchy–Schwarz inequality, assumption (6) and the fact that (u_n') is bounded in $L^2(0,T;H)$, there exists a constant $c_{13} > 0$ which is independent of n such that

$$\begin{aligned}
\left|\int_0^T \langle u_n', u_n - u\rangle_{g(t,u_n)}\,dt\right| &\leqslant \left(\int_0^T \|u_n'\|^2_{g(t,u_n)}\,dt\right)^{\frac{1}{2}} \left(\int_0^T \|u_n - u\|^2_{g(t,u_n)}\,dt\right)^{\frac{1}{2}} \\
&\leqslant c_2^2 \left(\int_0^T \|u_n'\|^2_H\,dt\right)^{\frac{1}{2}} \left(\int_0^T \|u_n - u\|^2_H\,dt\right)^{\frac{1}{2}} \\
&\leqslant c_{13} \left(\int_0^T \|u_n - u\|^2_H\,dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Using (17), the preceding inequality implies

$$\int_0^T \langle u_n', u_n - u\rangle_{g(t,u_n)}\,dt \to 0.$$

By combining this convergence, (25) and (24), we deduce that

$$\int_0^T \langle u_n', u_n\rangle_{g(t,u_n)}\,dt \to \int_0^T \langle u', u\rangle_{g(t,u)}\,dt.$$

This convergence, (23), (22) and equality (21) yield

$$\int_0^T E'(u_n)u_n\,dt \to \int_0^T \langle \chi, u\rangle_{V',V}\,dt. \tag{26}$$

We have the following equality

$$\int_0^T E'_\omega(u_n)u_n\,dt = \int_0^T E'(u_n)u_n\,dt + \omega\int_0^T \|u_n\|_H^2\,dt.$$

This implies after using (17) and (26)

$$\begin{aligned}\int_0^T E'_\omega(u_n)u_n\,dt &\to \int_0^T \langle \chi, u\rangle_{V',V}\,dt + \omega\int_0^T \langle u, u\rangle_H\,dt\\ &= \int_0^T \langle \chi + \omega u, u\rangle_{V',V}\,dt.\end{aligned} \tag{27}$$

Let $v \in L^\infty(0,T;V)$ and $\lambda \in \mathbb{R}$. By applying Lemma 9 to the function E_ω and by integrating over $(0,T)$ we have

$$\int_0^T \langle E'_\omega(u_n), u_n - u - \lambda v\rangle_{V',V}\,dt \geqslant \int_0^T \langle E'_\omega(u+\lambda v), u_n - u - \lambda v\rangle_{V',V}\,dt.$$

Letting $n \to \infty$ in this last inequality, we obtain after using (16), (18) and (27) that

$$\int_0^T \langle \chi + \omega u, \lambda v\rangle_{V',V}\,dt \leqslant \int_0^T \langle E'_\omega(u+\lambda v), \lambda v\rangle_{V',V}\,dt.$$

We divide by $\lambda > 0$, let $\lambda \to 0^+$, and we use the continuity of E' in order to obtain

$$\int_0^T \langle \chi, v\rangle_{V',V}\,dt \leqslant \int_0^T \langle E'(u), v\rangle_{V',V}\,dt.$$

Since $v \in L^\infty(0,T;V)$ is arbitrary, this implies that

$$E'(u) = \chi.$$

Then equality (20) becomes for every $v \in L^2(0,T;V)$

$$\int_0^T \langle u', v\rangle_{g(t,u)}\,dt + \int_0^T E'(u)v\,dt = \int_0^T \langle f(t), v\rangle_{g(t,u)}\,dt.$$

This implies that the function u satisfies the evolution equation of system (4).

Finally, we check that the function u satisfies the initial condition of system (4). Since the point evaluation in 0 from $W^{1,2}(0,T;H)$ into H is bounded and linear, it maps weakly convergent sequences into weakly convergent sequences, one has $u_n(0) \rightharpoonup u(0)$ in H. Since $u_n(0) = u_0^n \to u_0$ in V by the choice of (u_0^n), we obtain that $u(0) = u_0$. $\square$

4. Applications

Example 1. Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $1 < p < \infty$ such that $p > \frac{2N}{N+2}$. Let $\varepsilon \in (0,1)$ and let

$$m : [0,T] \times \Omega \times \mathbb{R} \to \left[\varepsilon, \frac{1}{\varepsilon}\right]$$

be a measurable function such that $m(t,x,\cdot)$ is continuous for every $(t,x) \in [0,T] \times \Omega$.

We consider the diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(t,\cdot,u)\,\Delta_p u = f & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \end{cases} \tag{28}$$

where Δ_p is the p-Laplace operator. This equation can be rewritten as a gradient system.

We put

$$V = W_0^{1,p}(\Omega),$$

which is a reflexive and separable Banach space for the norm

$$\|u\|_V = \|\nabla u\|_{L^p(\Omega)^N}.$$

Let $E : V \to \mathbb{R}$ be the function defined for every $u \in V$ by

$$E(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx.$$

We let

$$H = L^2(\Omega),$$

equipped with the usual inner product and norm.

By the Fubini–Lebesgue theorem, for every $u \in W_0^{1,p}(\Omega)$, there exists a set $N_u \subset (0,T)$ of measure zero such that for every $t \in (0,T) \setminus N_u$, the function $\frac{1}{m(t,\cdot,u(\cdot))}$ is measurable on Ω. Since $W_0^{1,p}(\Omega)$ is separable and since $\frac{1}{m}$ is continuous with respect to the third variable, we may construct a set $N \subset (0,T)$ of measure zero such that for every $t \in (0,T) \setminus N$ and every $u \in W_0^{1,p}(\Omega)$, the function $\frac{1}{m(t,\cdot,u(\cdot))}$ is measurable on Ω. We may therefore consider the function $g : (0,T) \times V \to \mathrm{Inner}(H)$ defined for every $t \in (0,T) \setminus N$, every $u \in W_0^{1,p}(\Omega)$ and every $v, w \in H$ by

$$\langle v, w \rangle_{g(t,u)} = \int_\Omega v w \frac{dx}{m(t,x,u(x))}.$$

We note that g is only defined for almost every $t \in (0,T)$; for $t \in N$, one might set $\langle\cdot,\cdot\rangle_{g(t,u)} = \langle\cdot,\cdot\rangle_{L^2(\Omega)}$.

Define the p-Laplace operator with Dirichlet boundary conditions on $L^2(\Omega)$ by

$$D(\Delta_p) = \left\{u \in W_0^{1,p}(\Omega)\colon \exists w \in L^2(\Omega),\ \forall v \in W_0^{1,p}(\Omega),\ \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx = -\int_\Omega wv\, dx\right\},$$

$$\Delta_p u = w.$$

With this definition we have for every $u \in D(\Delta_p)$, $v \in V$ and for almost every $t \in (0,T)$

$$E'(u)v = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx = -\int_\Omega (\Delta_p u)v\, dx = \big\langle -m(t,\cdot,u)\, \Delta_p u, v\big\rangle_{g(t,u)},$$

that is, $u \in D(\nabla_{g(t)}E)$ and

$$\nabla_{g(t)}E(u) = -m(t,\cdot,u)\, \Delta_p u.$$

Similarly, one proves that $D(\nabla_{g(t)}E) \subseteq D(\Delta_p)$, and hence $D(\nabla_{g(t)}E) = D(\Delta_p)$.

Corollary 10. *For every $f \in L^2(0,T;L^2(\Omega))$ and every $u_0 \in W_0^{1,p}(\Omega)$, problem (28) admits a solution $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W_0^{1,p}(\Omega))$ such that $u(t) \in D(\Delta_p)$ for almost every $t \in (0,T)$.*

Proof. It suffices to check that the assumptions of Theorem 4 are satisfied. Since $p > \frac{2N}{N+2}$, we obtain by the Rellich–Kondrachov theorem [5, Théorème IX.16, p. 169] that $W_0^{1,p}(\Omega)$ is compactly embedded into $L^2(\Omega)$. The function E_ω is coercive and convex for all $\omega \geqslant 0$, so that E is an H-elliptic function. We have also by Hölder's inequality, for every $R \geqslant 0$, and for every $u, v \in V$ such that $\|u\|_V \leqslant R$

$$\begin{aligned}\big|E'(u)v\big| &= \left|\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx\right| \leqslant \int_\Omega |\nabla u|^{p-1}|\nabla v|\, dx \leqslant \|\nabla u\|_{L^p(\Omega)^N}^{p-1}\|\nabla v\|_{L^p(\Omega)^N} \\ &= \|u\|_V^{p-1}\|v\|_V \leqslant R^{p-1}\|v\|_V.\end{aligned}$$

This proves that the derivative $E' : V \to V'$ maps bounded sets into bounded sets.

Let $(u_n) \subset V$ be such that $u_n \rightharpoonup u$ in V. Then (u_n) is bounded in V and by using the compact embedding $V \hookrightarrow H$ we can extract from (u_n) a subsequence (which we denote again (u_n)) such that $u_n \to u$ in H. This implies, after extracting a sequence again, that $u_n(x) \to u(x)$ for a.e. $x \in \Omega$. Using the Lebesgue dominated theorem, we deduce that $g(t,\cdot)$ is a metric for every $t \in [0,T]$.

For every $u \in W_0^{1,p}(\Omega)$ and every $v, w \in L^2(\Omega)$, the function $\frac{1}{m(\cdot,\cdot,u(\cdot))}vw$ is integrable on $(0,T) \times \Omega$, then by the Fubini–Lebesgue theorem the function $\langle v,w\rangle_{g(\cdot,u)}$ is measurable on $(0,T)$. Since m takes values in $[\varepsilon, \frac{1}{\varepsilon}]$, we obtain for every $t \in [0,T]$ and for every $v \in H$

$$\sqrt{\varepsilon}\,\|v\|_H \leqslant \|v\|_{g(t,u)} \leqslant \frac{1}{\sqrt{\varepsilon}}\|v\|_H.$$

Hence, assumption (6) is satisfied.

We let

$$\begin{aligned}
&u_n \rightharpoonup u && \text{in } W^{1,2}\big(0,T;L^2(\Omega)\big),\\
&u_n \overset{w^*}{\to} u && \text{in } L^\infty\big(0,T;W_0^{1,p}(\Omega)\big),\\
&v_n \rightharpoonup v && \text{in } L^2\big(0,T;L^2(\Omega)\big), \quad \text{and}\\
&w \in L^2\big(0,T;L^2(\Omega)\big).
\end{aligned}$$

Since the embedding $C([0,T];H) \hookrightarrow L^2(0,T;H)$ is continuous, it follows from Lemma 8 that the embedding

$$W^{1,2}\big(0,T;L^2(\Omega)\big) \cap L^\infty\big(0,T;W_0^{1,p}(\Omega)\big) \hookrightarrow L^2\big(0,T;L^2(\Omega)\big)$$

is compact. Hence we can extract from (u_n) a subsequence (which we denote again by (u_n)) such that

$$u_n \to u \quad \text{in } L^2\big(0,T;L^2(\Omega)\big).$$

Then we get (after passing to a subsequence again)

$$u_n(t,x) \to u(t,x) \quad \text{for a.e. } (t,x) \in (0,T) \times \Omega.$$

Since m is continuous with respect to the third variable and bounded away from 0, this implies

$$\frac{1}{m(t,x,u_n(t,x))} \to \frac{1}{m(t,x,u(t,x))} \quad \text{for a.e. } (t,x) \in (0,T) \times \Omega.$$

By the dominated convergence theorem, we obtain

$$\frac{1}{m(\cdot,\cdot,u_n)} w \to \frac{1}{m(\cdot,\cdot,u)} w \quad \text{in } L^2\big(0,T;L^2(\Omega)\big).$$

Hence

$$\int_0^T \langle v_n, w\rangle_{g(t,u_n)}\, dt \to \int_0^T \langle v, w\rangle_{g(t,u)}\, dt.$$

This proves the continuity assumption (7) and the claim follows from Theorem 4. $\square$

Example 2. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^1 and let ε, m and p be like in Example 1.

We consider the diffusion equation

$$\begin{cases}
\dfrac{\partial u}{\partial t} - m(t,\cdot,u)\,\Delta_p u = f & \text{in } (0,T) \times \Omega,\\
|\nabla u|^{p-2}\nabla u \cdot \vec{\nu} = 0 & \text{on } (0,T) \times \partial\Omega,\\
u(0,\cdot) = u_0 & \text{in } \Omega,
\end{cases} \tag{29}$$

where $\vec{\nu}$ is the outer unit normal to the boundary $\partial\Omega$.

We put

$$V = W^{1,p}(\Omega),$$

which is a reflexive and separable Banach space for the norm

$$\|u\|_V = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)^N}.$$

We put further

$$H = L^2(\Omega).$$

Let $E : V \to \mathbb{R}$ be the function defined for every $u \in V$ by

$$E(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx,$$

and let $g : (0, T) \times V \to \mathrm{Inner}(H)$ be the function defined for every $u \in V$, every $v, w \in H$ and for almost every $t \in (0, T)$ by

$$\langle v, w \rangle_{g(t,u)} = \int_{\Omega} v w \frac{dx}{m(t, x, u)}.$$

We define further the p-Laplace operator with Neumann boundary conditions on $L^2(\Omega)$ by

$$D(\Delta_p) = \left\{ u \in W^{1,p}(\Omega) \colon \exists w \in L^2(\Omega),\ \forall v \in W^{1,p}(\Omega),\ \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = -\int_{\Omega} w v \, dx \right\},$$

$$\Delta_p u = w.$$

Note that the Neumann type boundary condition $|\nabla u|^{p-2} \nabla u \cdot \vec{\nu} = 0$ is satisfied in a weak sense for every $u \in D(\Delta_p)$. In fact, if $u \in C^1(\overline{\Omega}) \cap D(\Delta_p)$ is such that $|\nabla u|^{p-2} \nabla u \in C^1(\overline{\Omega})$, then an integration by parts shows that $|\nabla u|^{p-2} \nabla u \cdot \vec{\nu} = 0$ on the boundary.

With the above definition of the p-Laplace operator we obtain like in Example 1 that $D(\nabla_{g(t)} E) = D(\Delta_p)$ and for every $u \in D(\Delta_p)$, for almost every $t \in (0, T)$

$$\nabla_{g(t)} E(u) = -m(t, \cdot, u) \, \Delta_p u.$$

Corollary 11. *For every $f \in L^2(0, T; L^2(\Omega))$ and every $u_0 \in W^{1,p}(\Omega)$, problem* (29) *admits a solution $u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(0, T; W^{1,p}(\Omega))$ such that $u(t) \in D(\Delta_p)$ for almost every $t \in (0, T)$.*

Proof. We check only that E is an H-elliptic function, the other assumptions of Theorem 4 are verified like in Corollary 10.

First, there exists a constant $C > 0$ such that for every $u \in V$ one has

$$\|u\|_{L^p(\Omega)} \leqslant C\big(\|\nabla u\|_{L^p(\Omega)^N} + \|u\|_{L^2(\Omega)}\big). \tag{30}$$

In fact, if $p \leqslant 2$, inequality (30) is clearly satisfied since the embedding $L^2(\Omega) \hookrightarrow L^p(\Omega)$ is continuous. If $p > 2$, we have the following embedding

$$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^2(\Omega),$$

where the first embedding is compact and the second is continuous. By [17, Lemma 1.1, p. 106], for every $\delta > 0$, there exists $C_\delta > 0$ such that for every $u \in V$ one has

$$\|u\|_{L^p(\Omega)} \leqslant \delta\big(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)^N}\big) + C_\delta \|u\|_{L^2(\Omega)}.$$

Hence, inequality (30) follows by choosing $\delta < 1$ in this last inequality. Inequality (30) implies that for every $\omega > 0$ the function E_ω is coercive. $\square$

Example 3. Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let $\varepsilon \in (0,1)$ and let

$$a : \Omega \times L^2(\Omega) \to \left[\varepsilon, \frac{1}{\varepsilon}\right]$$

be a function such that

(a) $a(\cdot, u)$ is measurable for every $u \in L^2(\Omega)$,
(b) $a(x, \cdot)$ maps weakly convergent sequences in $L^2(\Omega)$ into convergent sequences in $\mathbb{R}$ for almost every $x \in \Omega$.

Consider the following evolution equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\big(a(\cdot, u)\nabla u\big) = f & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega. \end{cases} \tag{31}$$

Eq. (31) can be rewritten as a gradient system. To see this, we put

$$V = L^2(\Omega)$$

and

$$H = H^{-1}(\Omega).$$

We consider the bounded and coercive inner products $l : L^2(\Omega) \to \operatorname{Inner}(H_0^1(\Omega))$ defined for every $u \in L^2(\Omega)$ and every $v, w \in H_0^1(\Omega)$ by

$$\langle v, w\rangle_{l_u} = \int_\Omega a(x,u)\nabla v(x) \cdot \nabla w(x)\,dx.$$

By the Lax–Milgram theorem, the associated operators $L_u : H_0^1(\Omega) \to H^{-1}(\Omega)$, $L_u v := \langle v, \cdot\rangle_{l_u}$ are bounded and invertible. We denote by L_u^{-1} the inverses.

By integrating by parts, we obtain for every $u \in L^2(\Omega)$ and every $v \in H_0^1(\Omega)$

$$L_u v = -\operatorname{div}\big(a(x,u)\nabla v\big) \quad \text{in } \mathcal{D}'(\Omega).$$

Let $g : V \to \operatorname{Inner}(H)$ be the function defined for every $u \in V$, and every $v, w \in H$ by

$$\langle v, w\rangle_{g(u)} = \langle v, L_u^{-1} w\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Let $E : V \to \mathbb{R}$ be the function defined for every $u \in L^2(\Omega)$ by

$$E(u) = \frac{1}{2}\int_\Omega |u|^2\, dx.$$

Let $u \in D(\nabla_g E)$. Then there exists $w \in H^{-1}(\Omega)$ such that for every $v \in L^2(\Omega)$ one has

$$\int_\Omega uv\, dx = E'(u)v = \langle w, v\rangle_{g(t,u)} = \langle v, L_u^{-1} w\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega (L_u^{-1} w) v\, dx.$$

Then we obtain $u = L_u^{-1} w \in H_0^1(\Omega)$ and $\nabla_g E(u) = w = L_u u$.

Similarly one proves that $H_0^1(\Omega) \subset D(\nabla_g E)$ and hence, $D(\nabla_g E) = H_0^1(\Omega)$.

Corollary 12. *For every $f \in L^2(0, T; H^{-1}(\Omega))$ and every $u_0 \in L^2(\Omega)$, problem* (31) *admits a solution $u \in W^{1,2}(0, T; H^{-1}(\Omega)) \cap C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$.*

To prove Corollary 12, we need the following lemma. We omit the proof which is straightforward when using Lemma 8 and the fact that every function in $W^{1,2}(0, T; H^{-1}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ is weakly continuous with values in $L^2(\Omega)$.

Lemma 13. *Let (u_n) be a sequence such that*

$$u_n \rightharpoonup u \quad \text{in } W^{1,2}(0, T; H^{-1}(\Omega)) \quad \text{and}$$
$$u_n \overset{w^*}{\to} u \quad \text{in } L^\infty(0, T; L^2(\Omega)).$$

Then we have

$$u_n(t) \rightharpoonup u(t) \quad \text{in } L^2(\Omega) \text{ for every } t \in [0, T].$$

Proof of Corollary 12. In order to prove that there exists a solution u in the space $W^{1,2}(0, T; H^{-1}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, it suffices to check that the assumptions of Theorem 4 are satisfied. Since Ω is bounded, the embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is compact. The function E is clearly continuously differentiable, H-elliptic and the derivative $E' : V \to V'$ maps bounded sets into bounded sets.

Let $(u_n) \subset L^2(\Omega)$ such that $u_n \rightharpoonup u$ in $L^2(\Omega)$. By the continuity assumption on a, we obtain

$$a(x, u_n) \to a(x, u) \quad \text{for almost every } x \in \Omega. \tag{32}$$

Using the Cauchy–Schwarz inequality, we have for every $v, w \in H_0^1(\Omega)$ such that $\|w\|_{H_0^1(\Omega)} \leqslant 1$

$$\begin{aligned}\left|\langle v, w\rangle_{l_{u_n}} - \langle v, w\rangle_{l_u}\right| &\leqslant \int_\Omega \left|a(x, u_n) - a(x, u)\right| \nabla v \cdot \nabla w\, dx \\ &\leqslant \left(\int_\Omega \left|a(x, u_n) - a(x, u)\right|^2 |\nabla v|^2\, dx\right)^{\frac{1}{2}}.\end{aligned}$$

Using the convergence (32) and the dominated convergence theorem, this implies that for every $v \in H_0^1(\Omega)$

$$L_{u_n} v \to L_u v \quad \text{in } H^{-1}(\Omega). \tag{33}$$

By the Lax–Milgram theorem, one has for every $w \in H^{-1}(\Omega)$

$$\left\| L_u^{-1} w \right\|_{H_0^1(\Omega)} \leqslant \frac{1}{\varepsilon} \|w\|_{H^{-1}(\Omega)}. \tag{34}$$

This implies for every $w \in H^{-1}(\Omega)$

$$\begin{aligned} \left\| L_{u_n}^{-1} w - L_u^{-1} w \right\|_{H_0^1(\Omega)} &= \left\| L_{u_n}^{-1} (L_{u_n} - L_u) L_u^{-1} w \right\|_{H_0^1(\Omega)} \\ &\leqslant \left\| L_{u_n}^{-1} \right\| \left\| (L_{u_n} - L_u) L_u^{-1} w \right\|_{H^{-1}(\Omega)} \\ &\leqslant \frac{1}{\varepsilon} \left\| (L_{u_n} - L_u) L_u^{-1} w \right\|_{H^{-1}(\Omega)} \to 0 \quad (n \to \infty). \end{aligned} \tag{35}$$

This yields that g is a metric.

Let $v \in H^{-1}(\Omega)$. As a consequence of (34), one obtains

$$\|v\|_{g(u)}^2 = \left\langle v, L_u^{-1} v \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \leqslant \|v\|_{H^{-1}(\Omega)} \left\| L_u^{-1} v \right\|_{H_0^1(\Omega)} \leqslant \frac{1}{\varepsilon} \|v\|_{H^{-1}(\Omega)}^2. \tag{36}$$

It is easy to see that for every $w \in H_0^1(\Omega)$

$$\|L_u w\| \leqslant \frac{1}{\varepsilon} \|w\|_{H_0^1(\Omega)}.$$

Using this last estimate, one obtains for every $u \in L^2(\Omega)$ and every $v \in H^{-1}(\Omega)$, $w \in H_0^1(\Omega)$

$$\begin{aligned} \langle v, w \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} &= \left\langle v, L_u^{-1} L_u w \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \langle v, L_u w \rangle_{g(u)} \\ &\leqslant \|v\|_{g(u)} \|L_u w\|_{g(u)} \leqslant \|v\|_{g(u)} \|L_u w\|_{H^{-1}(\Omega)}^{\frac{1}{2}} \|w\|_{H_0^1(\Omega)}^{\frac{1}{2}} \leqslant \frac{1}{\sqrt{\varepsilon}} \|v\|_{g(u)}, \end{aligned}$$

if $\|w\|_{H_0^1(\Omega)} \leqslant 1$. We have thus proved that $\sqrt{\varepsilon} \|v\|_{H^{-1}(\Omega)} \leqslant \|v\|_{g(u)}$.

Hence, by combining this last estimate with (36), we obtain

$$\sqrt{\varepsilon}\, \|v\|_{H^{-1}(\Omega)} \leqslant \|v\|_{g(u)} \leqslant \frac{1}{\sqrt{\varepsilon}}\, \|v\|_{H^{-1}(\Omega)},$$

and assumption (6) of Theorem 4 is satisfied.

We let

$$\begin{aligned} &u_n \rightharpoonup u && \text{in } W^{1,2}\big(0, T; H^{-1}(\Omega)\big), \\ &u_n \overset{w^*}{\to} u && \text{in } L^\infty\big(0, T; L^2(\Omega)\big), \\ &v_n \rightharpoonup v && \text{in } L^2\big(0, T; H^{-1}(\Omega)\big), \quad \text{and} \\ &w \in L^2\big(0, T; H^{-1}(\Omega)\big). \end{aligned}$$

We want to prove assumption (7) of Theorem 4. More precisely, we want to prove that

$$\int_0^T \langle v_n, L_{u_n}^{-1} w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt \to \int_0^T \langle v, L_u^{-1} w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt.$$

Since by assumption we have $v_n \rightharpoonup v$ in $L^2(0,T;H^{-1}(\Omega))$, it suffices to prove that

$$L_{u_n}^{-1} w \to L_u^{-1} w \quad \text{in } L^2\big(0,T;H_0^1(\Omega)\big).$$

By Lemma 13, we have

$$u_n(t) \rightharpoonup u(t) \quad \text{in } L^2(\Omega) \text{ for every } t \in [0,T].$$

By (35), one has

$$L_{u_n}^{-1} w(t) \to L_u^{-1} w(t) \quad \text{in } H_0^1(\Omega) \text{ for a.e. } t \in (0,T).$$

Using the fact that $L_{u_n}^{-1}$ is uniformly bounded, the dominated convergence theorem for Bochner-integrable functions yields that assumption (7) in Theorem 4 is satisfied and we deduce from Theorem 4 that there exists $u \in W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ which is a solution of (31).

From system (31) one has for almost every $t \in (0,T)$

$$L_u u = f(t) - \frac{\partial u}{\partial t}.$$

This implies that for almost every $t \in (0,T)$

$$u = L_u^{-1}\left(f(t) - \frac{\partial u}{\partial t}\right).$$

Then we obtain

$$\|u(t)\|_{H_0^1(\Omega)} \leqslant \frac{1}{\varepsilon}\left(\|f(t)\|_{H^{-1}(\Omega)} + \left\|\frac{\partial u}{\partial t}(t)\right\|_{H^{-1}(\Omega)}\right)$$

and hence $u \in L^2(0,T;H_0^1(\Omega))$.

Using the fact that $W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ is a subspace of $C([0,T];L^2(\Omega))$ by [7, Theorem 3, p. 287], we deduce that $u \in C([0,T];L^2(\Omega))$. $\square$

Remark 14. Problem (31) was studied by A.A. Ovono and A. Rougirel [16] in an elliptic framework where the coefficient a is given by

$$a(x,u) = \beta\left(\int_{\Omega\cap B(x,r)} u(y)\,dy\right).$$

The function $\beta : \mathbb{R} \to \mathbb{R}$ is continuous bounded and verifies $\inf_{\mathbb{R}} \beta > 0$, and $B(x,r)$ denotes the open ball of center x and radius r.

We note that in this case, the coefficient a satisfies the weak continuity used in Example 3.

Remark 15. The choice of the energy E and the metric g in Example 3 is inspired by F. Otto's work [15] on the porous medium equation.

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